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An open problem on inverse matrices from industrial organization, and a partial solution

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ABSTRACT

This note formulates a large class of matrices whose inverses form an open research problem. It also provides a partial solution as a starting point to tackle the problem in future studies.

1. Introduction

The formation and stability of a coalition structure (or a partition) is an important and unsettled issue in both sciences and social sciences, such as artificial intelligence (Monderer and Tennenholtz [5]) and game theory (Shubik [8]). An obstacle in resolving this issue is that one needs to obtain the analytical expressions of equilibrium payoffs for an arbitrary coalition structure, which often requires one to invert matrices whose inverses are unknown. This note derives a large class of such matrices from industrial organization whose inverses yield the strategic equilibria (or Nash equilibria [6]) in most linear oligopoly models. Such inverses form an open problem, which includes *the analytical expression of the inverse for a general symmetric matrix.*

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As a starting point to find a complete solution in future studies, the paper provides a partial solution by inverting a non-trivial subset of the derived matrices. This partial solution, previously unavailable in the literature, will be useful to other scholars in their future studies.[1] The rest of the note is organized as follows: Section 2 defines and Section 3 derives the open problem, Sections 4 and 5 provide a partial solution and an application, Section 6 concludes, and the appendix provides proofs.

2. Description of the problem

Given a partition (or a coalition structure or a set of multi-product firms) $\Delta = \{S_1, S_2, \ldots, S_h\}$ of $N = \{1, \ldots, n\}$, let $n_i = |S_i|$ denote the cardinality of each coalition S_i (or the number of products in S_i, $1 \leqslant n_i \leqslant n$, $\sum_{i=1}^{h} n_i = n$), and h_1 the number of its singleton coalitions (i.e., those S_i with $n_i = 1$). Then, our main matrix B has $[n + h(h+1)/2 - h_1]$ constants distributed as below: there are h symmetric $n_i \times n_i$ submatrices B_{ii} on the main diagonal whose diagonal entries are a_k ($k \in S_i$) and off-diagonal entries are a constant $-b_i$ ($i = 1, \ldots, h$ and $n_i \geqslant 2$), and $h(h-1)$ other submatrices $B_{ij} = B_{ji}^{\top}$ with a dimension of $n_i \times n_j$ and an identical entry of $-c_{ij} = -c_{ji}$ ($j = 1, \ldots, h, j \neq i$). Precisely, the matrix B has the following structure:

$$B = B_{n\times n} = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1h} \\ B_{21} & B_{22} & \cdots & B_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ B_{h1} & B_{h2} & \cdots & B_{hh} \end{pmatrix}, \tag{1}$$

whose block submatrices B_{ii} and B_{ij} ($i, j = 1, \ldots, h, j \neq i$) are defined as

$$B_{ii} = \begin{pmatrix} a_{\hat{i}+1} & -b_i & \cdots & -b_i \\ -b_i & a_{\hat{i}+2} & \cdots & -b_i \\ \vdots & \vdots & \ddots & \vdots \\ -b_i & -b_i & \cdots & a_{\hat{i}+n_i} \end{pmatrix}_{n_i\times n_i} \quad \text{and} \quad B_{ij} = \begin{pmatrix} -c_{ij} & -c_{ij} & \cdots & -c_{ij} \\ -c_{ij} & -c_{ij} & \cdots & -c_{ij} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{ij} & -c_{ij} & \cdots & -c_{ij} \end{pmatrix}_{n_i\times n_j},$$

where $\hat{i} = \sum_{k=1}^{i-1} n_k$ is the number of rows (or columns) preceding B_{ii}; $a_k > 0$ ($k = 1, \ldots, n$), $b_i > 0$ ($i = 1, \ldots, h$ and $n_i \geqslant 2$), and $c_{ij} = c_{ji} > 0$ ($i, j = 1, \ldots, h$ and $j \neq i$) are $[n + h(h+1)/2 - h_1]$ constants whose economic interpretations are provided in (5) in the next section.[2]

As an example, the matrix for $\Delta = \{[1, 2], [3, 4], [5]\}$ with $S_1 = \{1, 2\}, S_2 = \{3, 4\}$, and $S_3 = \{5\}$ (i.e., $n = 5, h = 3, n_1 = 2, n_2 = 2, n_3 = 1$, and $h_1 = 1$) has $[n + h(h+1)/2 - h_1] = [5 + 3(3 + 1)/2 - 1] = 10$ constants and is given by

$$B_{5\times 5} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{pmatrix} a_1 & -b_1 & -c_{12} & -c_{12} & -c_{13} \\ -b_1 & a_2 & -c_{12} & -c_{12} & -c_{13} \\ -c_{12} & -c_{12} & a_3 & -b_2 & -c_{23} \\ -c_{12} & -c_{12} & -b_2 & a_4 & -c_{23} \\ -c_{13} & -c_{13} & -c_{23} & -c_{23} & a_5 \end{pmatrix}, \tag{2}$$

[1] A special case of this partial solution has already been used in several recent studies such as Ishibashi [4] and Wang and Zhao [9].

[2] Note that there will be no b_i in B_{ii} if $n_i = 1$ or S_i is a singleton. The example in (2) has no b_3 in B_{33}, because of $n_3 = 1$.

where the submatrices are:

$$B_{11} = \begin{bmatrix} a_1 & -b_1 \\ -b_1 & a_2 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} a_3 & -b_2 \\ -b_2 & a_4 \end{bmatrix}, \quad B_{33} = [a_5],$$

$$B_{12} = B_{21}^{\top} = \begin{bmatrix} -c_{12} & -c_{12} \\ -c_{12} & -c_{12} \end{bmatrix}, \quad B_{13} = B_{31}^{\top} = \begin{bmatrix} -c_{13} \\ -c_{13} \end{bmatrix}, \quad \text{and} \quad B_{23} = B_{32}^{\top} = \begin{bmatrix} -c_{23} \\ -c_{23} \end{bmatrix}.$$

Now, our open problem is to find the analytical expressions for the inverse of B in (1). Since an answer will include the analytical expression for the inverse of a general symmetric matrix (i.e., $h = h_1 = n$ with $n(n+1)/2$ constants), there might be no solution to the open problem. However, as implied in Sections 4 and 5, there remain large classes of partial solutions, to be discovered, that are both non-trivial and useful in future applications.

3. Derivation of the main matrix

This section first defines a multi-product linear Bertrand oligopoly.[3] It then shows that its equilibrium price has the form $p = B^{-1}d$, where B is the main matrix in (1) and d is a vector of constants determined by market parameters. Finally, it shows that the strategic equilibria in most, if not all, other linear oligopoly models also have the structure of B in (1).

A multi-product linear Bertrand oligopoly with n differentiated goods is defined by three parts: (1) n cost functions $C_k(q_k) = c_k q_k$, $k \in N = \{1, \ldots, n\}$; (2) a set of multi-product firms $H = \{1, \ldots, h\}$, with firm i producing $n_i = |S_i|$ products in $S_i \subseteq N$ ($1 \leqslant n_i \leqslant n$, $\sum_{j=1}^{h} n_j = n$), which is equivalent to a partition $\Delta = \{S_1, S_2, \ldots, S_h\}$ of N; and (3) demand functions for each firm's products given in (3) below.

Let p_k denote the price for each good $k \in N$. For each firm $S = S_i \in \Delta$, let $p_S = \{p_k |\, k \in S\}$ denote the vector of its prices and $p_{-S} = \{p_k |\, k \in N \backslash S\}$ the vector of other firms' prices. Given the industry's price $p = (p_1, \ldots, p_n)^{\top} = (p_S, p_{-S}) = \{p_S |\, S \in \Delta\}$, demands for each product $k \in S_i$ produced by a firm $S_i \in \Delta$ are

$$q_k(p) = q_k(p_{S_i}, p_{-S_i}) = V - \gamma_{kk} p_k + \gamma_i \Sigma_{m \in S_i \backslash \{k\}} p_m + \Sigma_{j \in H \backslash \{i\}} \gamma_{ij} \Sigma_{m \in S_j} p_m, \tag{3}$$

where $V > 0$ is demand size, and $\gamma_{kk} > 0$ ($k \in N$) and $0 < \gamma_i, \gamma_{ij} = \gamma_{ji} \leqslant 1$ ($i, j \in H, i \neq j$) are substitution parameters.[4] Now, the profit for each firm $S \in \Delta$ is equal to $\pi_S(p) = \pi_S(p_S, p_{-S}) = \Sigma_{k \in S}(p_k - c_k) q_k(p)$, with $q_k(p)$ as given in (3).

Strategic behavior assumes that each firm chooses a *best response*, or that it takes the prices of other firms as given and chooses its prices to maximize its profit. Precisely, a strategic equilibrium (or Bertrand–Nash equilibrium [1,6]) is a price vector $p^* = \{p_S^* |\, S \in \Delta\}$ such that p_S^* is a solution of *Max* $\{\pi_S(p_S, p_{-S}^*) |\, p_S \geqslant 0\}$ for each $S \in \Delta$. Under usual assumptions (e.g., a unique solution exists), such equilibrium price solves the following h sets of first order conditions:

$$\frac{\partial \pi_{S_i}(p)}{\partial p_k} = 0, \quad \text{all } k \in S_i \text{ and for each } S_i \in \Delta, \text{ or } Bp = d, \tag{4}$$

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[3] Bertrand oligopoly [1] (or oligopoly under price competition) assumes that each firm's choices are the prices of its products. In contrast, Cournot oligopoly [2] (or oligopoly under quantity competition) assumes that each firm's choices are the quantities of its products.

[4] Note that internal substitution within a firm i has identical rate γ_i (i.e., between any m and $t \in S_i$), and external substitution between two firms $i \neq j$ has identical rate γ_{ij} among all of their products (i.e., between any $m \in S_i$ and $t \in S_j$).

where B is the same as in (1) with[5]

$$a_k = 2\gamma_{kk}, \quad b_i = 2\gamma_i, \quad c_{ij} = \gamma_{ij}, \quad d = \{d_{S_i} \mid S_i \in \Delta\} \text{ with}$$
$$d_{S_i} = \{d_k \mid k \in S_i\}, \quad \text{where } d_k = V + \gamma_{kk}c_k - \gamma_i \Sigma_{m \in S_i \setminus \{k\}} c_m \tag{5}$$

for all $k \in S_i$ and each $S_i \in \Delta$.

Below we discuss ten classes of important linear oligopoly models whose strategic equilibria all have the same structure of (1). Note that Bertrand equilibria with single-product firms (i.e., when $h = n$ in (3)) are often called premerger Bertrand equilibria. Because the most general single-product model (i.e., with $n(n-1)/2$ substitution parameters) remains as an open problem, most merger studies in Bertrand oligopoly use the simplified models of Shubik [7] (as in cases 1 and 2 below) or Dixit [3].

Case 1. *Simultaneous mergers without synergy*. Let $\pi_k(p) = (p_k - c_k)q_k(p)$ be firm k's profit, with $q_k(p)$ as the Shubik demand in (10). Now, let a partition $\Delta = \{S_1, S_2, \ldots, S_h\}$ of N denote a set of h simultaneous mergers from the premerger equilibrium $p^0 = \{p_i^0 \mid i \in N\}$ (i.e., $p_i^0 \in ArgMax\{\pi_i(p_i, p_{-i}^0) \mid p_i \geqslant 0\}$, all i), and assume that the profit function for each merger $S \in \Delta$ is

$$\pi_S(p) = \Sigma_{k \in S}(p_k - c_k)q_k(p). \tag{6}$$

Then, $p^* = B^{-1}d$ in (4) becomes a postmerger equilibrium without synergy.

Case 2. *Simultaneous mergers with weak cost-synergy*. This is the same as case 1 except that

$$\pi_S(p) = \Sigma_{k \in S}(p_k - c_S)q_k(p) \tag{7}$$

for each $S \in \Delta$, where $c_S = Min\{c_k \mid k \in S\}$. Note that cases 1 and 2 will become two new merger models when the demand in (6–7) is replaced by the Dixit demand in (9).

Case 3. *Simultaneous mergers with marketing-synergy*. Start with $h = n$ in (3), and assume that each merger $S \in \Delta = \{S_1, S_2, \ldots, S_h\}$ generates the following form of marketing-synergy: for each i and each $j \neq i \in H$, let

$$\gamma_i = Max\{\gamma_{km} \mid k \neq m \in S_i\}, \quad \text{and} \quad \gamma_{ij} = \gamma_{ji} \equiv Min\{\gamma_{km} \mid k \in S_i, m \in S_j\}, \tag{8}$$

which reduces (increases) the positive effects of a price increase on other firms' demand (demands for the firm's other products). Now, (3) becomes the demands for products sold by each merger $S \in \Delta$, and $p^* = B^{-1}d$ in (4) becomes a new postmerger equilibrium with marketing-synergy.

Case 4. *Dixit demand system* ([3], 1979):

$$q_k(p) = V - p_k + \gamma\, \Sigma_{m \neq k} p_m, \text{ all } k \in N. \tag{9}$$

This is a special case of (3) when $h = n$ (or there is no γ_i), $\gamma_{kk} \equiv 1$ and $\gamma_{ij} \equiv \gamma$ ($i \neq j, k \in N$), which follows from a simple utility maximization problem.[6] The solution $p^* = B^{-1}d$ in (4) now is a premerger equilibrium with single-product firms.

Case 5. *Shubik demand system* ([7], 1980):

$$q_k(p) = V - p_k - \gamma(p_k - \overline{p}), \text{ all } k \in N, \tag{10}$$

where $\overline{p} = (\Sigma p_m)/n$ is the average price. This is another special case of (3) ($h = n$, $\gamma_{kk} \equiv [n + (n-1)\gamma]/n$, $\gamma_{ij} \equiv \gamma/n$, $i \neq j, k \in N$). It yields a different premerger equilibrium with single-product firms, and it has the advantage of being intuitive: a firm k will be penalized (rewarded) by an amount equal to $\gamma|p_k - \overline{p}|$ when it charges more (less) than the average price or when $(p_k - \overline{p}) > 0\ (< 0)$.

[5] The above conditions can be rearranged as: for each $S_i \in \Delta$, and for all $k \in S_i$,

$$2\gamma_{kk}p_k - 2\gamma_i \Sigma_{m \in S_i \setminus \{k\}} p_m - \Sigma_{j \in H \setminus \{i\}} \gamma_{ij} \Sigma_{m \in S_j} p_m = V + \gamma_{kk}c_k - \gamma_i \Sigma_{m \in S_i \setminus \{k\}} c_m,$$

which leads to $Bp = d$, whose parameters are given in (5).

[6] Let $I_{n\times n}$ be the identity matrix, $E_{n\times n}$ the matrix of ones, $G = (1-\gamma)I_{n\times n} + \gamma E_{n\times n}$, and $U(q, y) = y + V\Sigma q_m - q^\top Gq/2$ the utility, where y is a composite measure of all other consumptions. Then, $Max\left\{U(q, y) \mid p^\top q + y \leqslant Y\right\}$ yields the inverse version of (9), where Y is fixed income.

Below we briefly show that strategic equilibria in most linear Cournot (or quantity-setting) oligopolies also have the structure of (1). Given a set of firms $H = \{1, \ldots, h\}$ or $\Delta = \{S_1, S_2, \ldots, S_h\}$, let $q = (q_S, q_{-S}) = \{q_S | S \in \Delta\}$ be the vector of all products,

$$p_k(q) = \widehat{V} - \widehat{\gamma}_{kk} q_k - \widehat{\gamma}_i \Sigma_{m \in S_i \setminus \{k\}} q_m - \Sigma_{j \in H \setminus \{i\}} \widehat{\gamma}_{ij} \Sigma_{m \in S_j} q_m, \quad \text{all } k \in S_i, \tag{11}$$

be the *inverse demand* for the products of each firm $S_i \in \Delta$ ($\widehat{\gamma}_{kk}, \widehat{\gamma}_i, \widehat{\gamma}_{ij} > 0, k \in N, i \neq j \in H$), $\pi_S(q_S, q_{-S}) = \Sigma_{k \in S}(p_k(q) - c_k) q_k$ be the profit of each $S \in \Delta$, and $q^* = \{q_S^* | S \in \Delta\}$ be the strategic equilibrium (or Cournot–Nash equilibrium [2,6]) such that each q_S^* is a *best response* to q_{-S}^* (i.e., $q_S^* \in ArgMax\{\pi_S(q_S, q_{-S}^*) | q_S \geqslant 0\}$). Then, under usual conditions, such equilibrium supply is uniquely determined by

$$\frac{\partial \pi_S(q_S, q_{-S})}{\partial q_k} = 0, \quad \text{for all } k \in S \text{ and each } S \in \Delta, \text{ or } \overline{B} q = \overline{d}, \tag{12}$$

where $\overline{B}$ has the same structure of B in (1). Cases 6–10 in (13) below are respectively the Cournot equivalent of cases 1–5 in equations (6–10):

$$\begin{aligned}
&\textit{Case } 6. \quad \pi_S(q) = \Sigma_{k \in S}(p_k(q) - c_k) q_k, \ \text{all } S \in \Delta; \\
&\textit{Case } 7. \quad \pi_S(q) = \Sigma_{k \in S}(p_k(q) - c_S) q_k, \ \text{all } S \in \Delta; \\
&\textit{Case } 8. \quad \widehat{\gamma}_i = Min\{\widehat{\gamma}_{km} | k \neq m \in S_i\}, \widehat{\gamma}_{ij} = \widehat{\gamma}_{ji} \equiv Max\{\widehat{\gamma}_{km} | k \in S_i, m \in S_j\}; \\
&\textit{Case } 9. \quad p_k(q) = \widehat{V} - q_k - \widehat{\gamma} \Sigma_{m \neq k} q_m, \ \text{all } k \in N; \\
&\textit{Case } 10. \quad p_k(q) = \widehat{V} - q_k + \widehat{\gamma}(q_k - \overline{q}), \ \text{all } k \in N,
\end{aligned} \tag{13}$$

where $\overline{q} = \sum q_j / n$ in case 10 is the average output, and synergy in case 8 reduces (increases) the negative effects of an output increase on own (other firms') profits.

The details of these five important classes of quantity-setting models are skipped as they are similar to the earlier price-setting model (3). It is useful to note that most Cournot models in (13) and Bertrand models in (6–10) can be inverted from each other. For example, inverting the Shubik system in (10) yields the inverse demand in case 10, with $\widehat{V} = V$ and $\widehat{\gamma} = \gamma/(1 + \gamma)$.

4. A partial solution [7]

Consider the class of B in (1) given by: $a_k \equiv a$, all k, $b_i \equiv b$ and $c_{ij} = c_{ji} \equiv c$, all $i \neq j$. Denote such matrices by

$$A = A_{n \times n} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1h} \\ A_{21} & A_{22} & \cdots & A_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ A_{h1} & A_{h2} & \cdots & A_{hh} \end{pmatrix}, \tag{14}$$

where

$$A_{ii} = \begin{pmatrix} a & -b & \cdots & -b \\ -b & a & \cdots & -b \\ \vdots & \vdots & \ddots & \vdots \\ -b & -b & \cdots & a \end{pmatrix}_{n_i \times n_i} \quad \text{and} \quad A_{ij} = \begin{pmatrix} -c & -c & \cdots & -c \\ -c & -c & \cdots & -c \\ \vdots & \vdots & \ddots & \vdots \\ -c & -c & \cdots & -c \end{pmatrix}_{n_i \times n_j}.$$

[7] I am indebted to Eric Howe for allowing me to report the main results of this section, which had been circulated in an unpublished note, titled "Merger incentives and inverse matrices from Bertrand competition," by E. Howe and J. Zhao (2004).

The analytic expressions of A^{-1} are given as a theorem below.[8]

Theorem 1. *Let $U = (u_{ij})_{n\times n} = A^{-1}$ denote the inverse of A given in (14), for $i = 1, \ldots, h$, define*

$$\beta_i = \left(n_i + \frac{a+(1-n_i)b}{c}\right)^{-1} = \frac{c}{a+b+(c-b)n_i};$$
$$\alpha = \sum_{i=1}^{h}\beta_i n_i = c\sum_{i=1}^{h}\frac{n_i}{a+b+(c-b)n_i}, \tag{15}$$
$$\theta_i = \frac{1}{1-\alpha}\left(\beta_i n_i + \frac{c}{b}\sum_{\substack{j=1\\ j\neq i}}^{h}\beta_j n_j\right) = \frac{1}{1-\alpha}\left(\beta_i n_i + \frac{c}{b}(\alpha - \beta_i n_i)\right), \text{ and}$$

assume $\alpha \neq 1$ and $a + b + (c - b)n_i \neq 0$, all i. Then, $U = A^{-1} =$

$$\frac{1}{a+b}I + \frac{1}{c(a+b)}\begin{pmatrix} b\beta_1(1+\theta_1)E_{n_1\times n_1} & \beta_1(c+b\theta_2)E_{n_1\times n_2} & \cdots & \beta_1(c+b\theta_h)E_{n_1\times n_h} \\ \beta_2(c+b\theta_1)E_{n_2\times n_1} & b\beta_2(1+\theta_2)E_{n_2\times n_2} & \cdots & \beta_2(c+b\theta_h)E_{n_2\times n_h} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_h(c+b\theta_1)E_{n_h\times n_1} & \beta_h(c+b\theta_2)E_{n_h\times n_2} & \cdots & b\beta_h(1+\theta_h)E_{n_h\times n_h} \end{pmatrix}, \tag{16}$$

where I is the identity matrix and $E_{n_i\times n_j}$ is an $n_i \times n_j$ matrix of 1's.

Although it is quite involved to verify $UA = AU = I$, it is easy to verify (16) by three cases of A: (1) $h = n$ or $n_i = 1$ for all i; (2) $h = 1$ or $n_1 = n$, and replace b by c; and (3) $1 < h < n$ and replace b by c, then using the simple matrix of (29), which is the inverse of D in (27) in appendix.

Applying $(c + b\theta_j) = (a + b)\beta_j/(1 - \alpha)$, A^{-1} in (16) can be rearranged as

$$A^{-1} = U = \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1h} \\ U_{21} & U_{22} & \cdots & U_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ U_{h1} & U_{h2} & \cdots & U_{hh} \end{pmatrix} =$$

$$\frac{1}{a+b}I + \frac{1}{c}\begin{pmatrix} \frac{b\beta_1(1+\theta_1)}{(a+b)}E_{n_1\times n_1} & \frac{\beta_1\beta_2}{(1-\alpha)}E_{n_1\times n_2} & \cdots & \frac{\beta_1\beta_h}{(1-\alpha)}E_{n_1\times n_h} \\ \frac{\beta_2\beta_1}{(1-\alpha)}E_{n_2\times n_1} & \frac{b\beta_2(1+\theta_2)}{(a+b)}E_{n_2\times n_2} & \cdots & \frac{\beta_2\beta_h}{(1-\alpha)}E_{n_2\times n_h} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_h\beta_1}{(1-\alpha)}E_{n_h\times n_1} & \frac{\beta_h\beta_2}{(1-\alpha)}E_{n_h\times n_2} & \cdots & \frac{b\beta_h(1+\theta_h)}{(a+b)}E_{n_h\times n_h} \end{pmatrix}, \tag{17}$$

where for each $i = 1, \ldots, h$ and all $j \neq i$, $j = 1, \ldots, h$,

$$U_{ii} = \frac{1}{a+b}I_{n_i\times n_i} + \frac{b\beta_i(1+\theta_i)}{c(a+b)}E_{n_i\times n_i},$$
$$U_{ij} = \frac{\beta_i(c+b\theta_j)}{c(a+b)}E_{n_i\times n_j} = \frac{\beta_i\beta_j}{c(1-\alpha)}E_{n_i\times n_j}.$$

By (17), A^{-1} is symmetric and has the same block structure of A.

8 Note that A^{-1} exists under the usual conditions of the underlying oligopoly models. For example, given the values of a, b and c in (5), it is easy to show $\alpha < 1/2$ and $a + b + (c - b)n_i > 2(1 + \gamma)$, which guarantee the conditions for the existence of A^{-1} in Theorem 1.

5. Applications: the premerger and postmerger Bertrand equilibria

Given the above expressions of A^{-1} in (16), it is straightforward, though often quite involved, to calculate the Bertrand equilibrium $p^* = B^{-1}d$ in (4) with three substitution rates (i.e., $\gamma_{kk} \equiv a/2$, $\gamma_i \equiv b/2$ and $\gamma_{ij} \equiv c$) or Cournot equilibrium $\overline{q} = \overline{B}^{-1}\overline{d}$ in (12) with three substitution rates. This section calculates, as an example of applications, the postmerger equilibria for $\Delta = \{S, t+1, \ldots, n\}$ with no synergy using both Dixit and Shubik demands in (9–10). Without loss of generality, assume $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_t$, so $c_S = c_1$; and assume Assumption $A0$ below, which guarantees positive outputs for all firms at both premerger and postmerger equilibria.

A0 (Assumption 0): *For each $S = \{1, \ldots, t\}$,*

$$\frac{nV + (n + (n-t)\gamma)\overline{c_S} + \gamma(n-t)\overline{c_{-S}}}{(2n + (2n-2t)\gamma)} > \overline{c_S} \tag{18}$$

holds, where $\overline{c_S} = \sum_{i\in S} c_i/t$ and $\overline{c_{-S}} = \sum_{i\notin S} c_i/(n-t)$.

We first provide the postmerger equilibrium for case 1 in (6), which uses the Shubik demand in (10).

Theorem 2. *Let p^* be the postmerger equilibrium for $S = \{1, \ldots, t\}$ with no synergy in (6), $\overline{c_S}$ and $\overline{c_{-S}}$ be given in (18). Then, for $k = 1, \ldots, t$ and $j = t+1, \ldots, n$,*

$$\begin{aligned} p_k^* &= \frac{n(2n(1+\gamma)-\gamma)V}{\omega_0} + \frac{\gamma^2 t\,(n-t)\,\overline{c_S}}{2\omega_0} + \frac{(n-t)\gamma(n(1+\gamma)-\gamma)\overline{c_{-S}}}{\omega_0} + \frac{c_k}{2}, \\ p_j^* &= \frac{n(2n(1+\gamma)-t\gamma)V}{\omega_0} + \frac{t\gamma\,(n(1+\gamma)-t\gamma)\,\overline{c_S}}{\omega_0} \\ &\quad + \frac{\gamma\,(n-t)\,(n(1+\gamma)-\gamma)\,(2n(1+\gamma)-t\gamma)\,\overline{c_{-S}}}{(2n(1+\gamma)-\gamma)\omega_0} + \frac{(n(1+\gamma)-\gamma)c_j}{2n(1+\gamma)-\gamma}, \end{aligned} \tag{19}$$

where $\omega_0 = \gamma^2\,(n-t)\,(t+2n-2) + 2n\gamma\,(3n-t-1) + 4n^2$.

Now, let $t = 1$ and $c_i = c_1 (i \in S)$ in (19) respectively, one gets the premerger equilibrium and postmerger equilibrium with weak cost-synergy as given below:

Corollary 1. *Let p^{**} be the postmerger equilibrium for $S = \{1, \ldots, t\}$ with weak cost-synergy, p^0 be the premerger equilibrium, and $\bar{c} = (\sum_{j=1}^n c_j)/n$. Then, for $k = 1, \ldots, t$, $j = t+1, \ldots, n$, and all i,*

$$\begin{aligned} p_k^{**} &= \frac{n(2n(1+\gamma)-\gamma)V}{\omega_0} + \frac{(n-t)\gamma(n(1+\gamma)-\gamma)\overline{c_{-S}}}{\omega_0} + \frac{(\gamma^2 t\,(n-t)+\omega_0)c_1}{2\omega_0}, \\ p_j^{**} &= \frac{n(2n(1+\gamma)-t\gamma)V}{\omega_0} + \frac{t\gamma\,(n(1+\gamma)-t\gamma)\,c_1}{\omega_0} \\ &\quad + \frac{\gamma\,(n-t)\,(n(1+\gamma)-\gamma)\,(2n(1+\gamma)-t\gamma)\,\overline{c_{-S}}}{(2n(1+\gamma)-\gamma)\omega_0} + \frac{(n(1+\gamma)-\gamma)c_j}{2n(1+\gamma)-\gamma}; \\ p_i^0 &= \frac{n(V-\bar{c})}{n(2+\gamma)-\gamma} + \frac{n\,(\gamma+1)\,\bar{c}}{2n(1+\gamma)-\gamma} + \frac{(n(1+\gamma)-\gamma)c_i}{2n(1+\gamma)-\gamma}, \end{aligned} \tag{20}$$

where ω_0, $\overline{c_S}$ and $\overline{c_{-S}}$ are the same as that in Theorem 2.

Next, we provide the postmerger equilibrium for case 1 using Dixit demand in (9).

Theorem 3. *Let p^{D*} be the postmerger equilibrium for $S = \{1, \dots, t\}$ with no synergy in (6) using Dixit demand in (9). Then, for $k = 1, \dots, t$ and $j = t + 1, \dots, n$,*

$$p_k^{D*} = \frac{(\gamma + 2)\, V}{\omega_0^D} + \frac{t\,(n - t)\,\gamma^2 \overline{c_S}}{2\omega_0^D} + \frac{\gamma\,(n - t)\,\overline{c_{-S}}}{\omega_0^D} + \frac{c_k}{2}, \tag{21}$$

$$p_j^{D*} = \frac{(2 - (t - 2)\gamma)\, V}{\omega_0^D} + \frac{\gamma t\,(1 - (t - 1)\gamma)\,\overline{c_S}}{\omega_0^D} + \frac{\gamma\,(n - t)\,(2 - (t - 2)\,\gamma)\,\overline{c_{-S}}}{(2 + \gamma)\,\omega_0^D} + \frac{c_j}{2 + \gamma},$$

where $\omega_0^D = [t(n - t) - 2(n - 1)]\,\gamma^2 - 2\,(n + t - 3)\,\gamma + 4$.

Finally, let $t = 1$ and $c_i = c_1 (i \in S)$ in (21) respectively, one gets the premerger equilibrium and postmerger equilibrium with weak cost-synergy as given below:

Corollary 2. *Let p^{D**} be the postmerger equilibrium for $S = \{1, \dots, t\}$ with weak cost-synergy and p^{D0} the premerger equilibrium using Dixit demand in (9). Then, for $k = 1, \dots, t$, $j = t + 1, \dots, n$, and all i,*

$$p_k^{D**} = \frac{(\gamma + 2)\, V}{\omega_0^D} + \frac{\gamma\,(n - t)\,\overline{c_{-S}}}{\omega_0^D} + \frac{\left[(t\,(n - t) - (n - 1))\,\gamma^2 - (t + n - 3)\,\gamma + 2\right] c_1}{\omega_0^D},$$

$$p_j^{D**} = \frac{(2 - (t - 2)\gamma)\, V}{\omega_0^D} + \frac{\gamma t\,(1 - (t - 1)\gamma)\, c_1}{\omega_0^D} + \frac{\gamma\,(n - t)\,(2 - (t - 2)\,\gamma)\,\overline{c_{-S}}}{(2 + \gamma)\,\omega_0^D} + \frac{c_j}{2 + \gamma};$$

$$p_i^{D0} = \frac{V - \bar{c}}{2 - (n - 1)\,\gamma} + \frac{(2 + (n + 1)\,\gamma)\,\bar{c}}{(2 + \gamma)\,(2 - (n - 1)\,\gamma)} + \frac{c_i}{2 + \gamma}, \tag{22}$$

where ω_0^D is the same as that in Theorem 3.

6. Conclusion and discussion

The above analysis has formulated a large class of matrices whose inverses yield the strategic equilibria in most linear oligopoly models. A complete understanding of these matrices remains unknown and their properties form a long list of open problems: the analytical expressions of their inverses, their eigenvalues and eigenvectors, their rank correction properties, and how these properties are determined by the partitions.

The author hopes that readers will be motivated and challenged to investigate these open problems, in particular, to obtain more partial solutions or calculate more nontrivial sets of the inverses based on a particular set of models in industrial organization or networks, or on a particular set of partitions, or a particular set of parameters, which might someday lead to a complete answer in the future.

Appendix. Proofs

Our proof of Theorem 1 uses the following Lemma 1, which is also used to verify the main inverse. Let $S = (s_{ij})_{h \times h} = F^{-1}$ denote the inverse of the following matrix

$$F = (f_{ij})_{h \times h} = \begin{pmatrix} a_1 & -n_2 & \cdots & -n_h \\ -n_1 & a_2 & \cdots & -n_h \\ \vdots & \vdots & \ddots & \vdots \\ -n_1 & -n_2 & \cdots & a_h \end{pmatrix}, \tag{23}$$

where for each $j = 1, \dots, h$, $f_{jj} = a_j > 0$, and $f_{ij} = -n_j < 0$ for all $i \neq j$.

Lemma 1. *Given the above F in (23), define* $\alpha = \sum_{i=1}^{h}(n_i/(a_i+n_i))$ *and suppose* $\alpha \neq 1$. *Then, the inverse of F is given by S =*

$$\begin{pmatrix} \frac{1}{a_1+n_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{a_h+n_h} \end{pmatrix} + \frac{1}{(1-\alpha)} \begin{pmatrix} \frac{n_1}{(a_1+n_1)^2} & \frac{n_2}{(a_1+n_1)(a_2+n_2)} & \cdots & \frac{n_h}{(a_1+n_1)(a_h+n_h)} \\ \frac{n_1}{(a_2+n_2)(a_1+n_1)} & \frac{n_2}{(a_2+n_2)^2} & \cdots & \frac{n_h}{(a_2+n_2)(a_h+n_h)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n_1}{(a_h+n_h)(a_1+n_1)} & \frac{n_2}{(a_h+n_h)(a_2+n_2)} & \cdots & \frac{n_h}{(a_h+n_h)^2} \end{pmatrix}, \tag{24}$$

where

$$s_{ii} = \frac{1}{a_i+n_i} + \frac{n_i}{(1-\alpha)(a_i+n_i)^2}, \quad \textit{for all } i; \tag{25}$$

$$s_{ij} = \frac{n_j}{(1-\alpha)(a_i+n_i)(a_j+n_j)}, \quad \textit{for all } i \neq j. \tag{26}$$

Proof of Lemma 1. Let e_h be a h-dimensional column vector of 1's, $X_{h\times 1} = e_h$, $Y_{1\times h} = (-n_1, \ldots, -n_h)$, and $B_{h\times h}$ be a diagonal matrix with $b_{ii} = a_i + n_i$, all i. By $F = B + XY$ and the Sherman–Morrison–Woodbury formula, $F^{-1} = B^{-1} - B^{-1}X(I+YB^{-1}X)^{-1}YB^{-1}$. Using $(I+YB^{-1}X)^{-1} = 1/(1-\alpha)$, one gets $B^{-1}X(I+YB^{-1}X)^{-1}YB^{-1} =$

$$\frac{-1}{(1-\alpha)} \begin{pmatrix} \frac{n_1}{(a_1+n_1)^2} & \frac{n_2}{(a_1+n_1)(a_2+n_2)} & \cdots & \frac{n_h}{(a_1+n_1)(a_h+n_h)} \\ \frac{n_1}{(a_2+n_2)(a_1+n_1)} & \frac{n_2}{(a_2+n_2)^2} & \cdots & \frac{n_h}{(a_2+n_2)(a_h+n_h)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n_1}{(a_h+n_h)(a_1+n_1)} & \frac{n_2}{(a_h+n_h)(a_2+n_2)} & \cdots & \frac{n_h}{(a_h+n_h)^2} \end{pmatrix},$$

which leads to (24). □

Letting $h = n$ and $n_i = c$ for all i in (23–24) leads to the corollary below, a special case of which is used in verifying the main inverse in (16).

Corollary 3. *Let* $R = (r_{ij})_{n\times n} = D^{-1}$ *denote the inverse of D given by*

$$D = (d_{ij})_{n\times n} = \begin{pmatrix} a_1 & -c & \cdots & -c \\ -c & a_2 & \cdots & -c \\ \vdots & \vdots & \ddots & \vdots \\ -c & -c & \cdots & a_n \end{pmatrix}, \tag{27}$$

where $d_{ii} = a_i$, $d_{ij} = -c$, *all* $i \neq j$. *Suppose* $\alpha = \sum_{i=1}^{n} c/(a_i+c) \neq 1$. *Then, R =*

$$\begin{pmatrix} \frac{1}{a_1+c} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{a_n+c} \end{pmatrix} + \frac{c}{(1-\alpha)} \begin{pmatrix} \frac{1}{(a_1+c)^2} & \frac{1}{(a_1+c)(a_2+c)} & \cdots & \frac{1}{(a_1+c)(a_n+c)} \\ \frac{1}{(a_2+c)(a_1+c)} & \frac{1}{(a_2+c)^2} & \cdots & \frac{1}{(a_2+c)(a_n+c)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(a_n+c)(a_1+c)} & \frac{1}{(a_n+c)(a_2+c)} & \cdots & \frac{1}{(a_n+c)^2} \end{pmatrix}, \tag{28}$$

where

$$r_{ii} = \frac{1}{a_i + c} + \frac{c}{(1-\alpha)(a_i+c)^2}, \quad \textit{for all } i;$$

$$r_{ij} = \frac{c}{(1-\alpha)(a_i+c)(a_j+c)}, \quad \textit{for all } i \neq j.$$

When $a_i = a$ for all i, the above inverse matrix becomes

$$D^{-1} = \frac{1}{a+c}\left\{I + \frac{c}{a-(h-1)c}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\right\} = \frac{1}{a+c}G = \frac{1}{a+c}\{g_{ij}\}_{h\times h}, \tag{29}$$

where $g_{ii} = [a-(h-2)c]\,/\,[a-(h-1)c]$, $g_{ij} = c/\,[a-(h-1)c]$, all $j \neq i$, which is useful to verify the main result of (16). We are now ready to prove the main theorem.

Proof of Theorem 1. Define $B_{n\times n} = (a+b)I_{n\times n}$,where $I_{n\times n}$ is the $n \times n$ identity matrix. For each $i = 1, \ldots, h$, define $X_i = (-b)e_{n_i}$,where e_h is a h-dimensional column vector of 1's; $Y_i = e_{n_i}^\top$, $Y_{ij} = \frac{c}{b}e_{n_j}^\top$ for all $j \neq i$, and

$$X_{n\times h} = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_h \end{pmatrix}, \quad Y_{h\times n} = \begin{pmatrix} Y_1 & Y_{12} & \cdots & Y_{1h} \\ Y_{21} & Y_2 & \cdots & Y_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{h1} & Y_{h2} & \cdots & Y_h \end{pmatrix},$$

where the 0's are column vector of zeros with the appropriate dimensions. Then, one sees $A = B + XY$, so A is a rank h adjustment of B. In order to compute $A^{-1} = B^{-1} - B^{-1}X(I + YB^{-1}X)^{-1}YB^{-1}$, one needs to obtain F^{-1} with

$$F = I + YB^{-1}X = \frac{c}{a+b}\begin{pmatrix} \frac{a+(1-n_1)b}{c} & -n_2 & \cdots & -n_h \\ -n_1 & \frac{a+(1-n_2)b}{c} & \cdots & -n_h \\ \vdots & \vdots & \ddots & \vdots \\ -n_1 & -n_2 & \cdots & \frac{a+(1-n_h)b}{c} \end{pmatrix}_{h\times h}.$$

Now, using the formula for F^{-1} in Lemma 1 or in (24–26), one has

$$F^{-1} = \frac{a+b}{c}\left\{\begin{pmatrix} \beta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_h \end{pmatrix} + \frac{1}{1-\alpha}\begin{pmatrix} n_1\beta_1^2 & n_2\beta_1\beta_2 & \cdots & n_h\beta_1\beta_h \\ n_1\beta_2\beta_1 & n_2\beta_2^2 & \cdots & n_h\beta_2\beta_h \\ \vdots & \vdots & \ddots & \vdots \\ n_1\beta_h\beta_1 & n_2\beta_h\beta_2 & \cdots & n_h\beta_h^2 \end{pmatrix}\right\}.$$

Note that $B^{-1}X = \frac{1}{a+b}X$ and $YB^{-1} = \frac{1}{a+b}Y$. Thus

$$B^{-1}XF^{-1}YB^{-1} = \frac{-1}{c(a+b)} \begin{pmatrix} b\beta_1(1+\theta_1)E_{n_1\times n_1} & \beta_1(c+b\theta_2)E_{n_1\times n_2} & \cdots & \beta_1(c+b\theta_h)E_{n_1\times n_h} \\ \beta_2(c+b\theta_1)E_{n_2\times n_1} & b\beta_2(1+\theta_2)E_{n_2\times n_2} & \cdots & \beta_2(c+b\theta_h)E_{n_2\times n_h} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_h(c+b\theta_1)E_{n_h\times n_1} & \beta_h(c+b\theta_2)E_{n_h\times n_2} & \cdots & b\beta_h(1+\theta_h)E_{n_h\times n_h} \end{pmatrix},$$

which leads to the formula in (16). $\square$

The proof of Theorem 2 involves the following special case of A in (14) with $\Delta = \{S, t+1, \ldots, n\}$ (i.e., $n_1 = t$ and $n_j = 1$ for all $j = 2, \ldots, h = n - t + 1$):

$$A = A_{n\times n} = \begin{pmatrix} a & \cdots & -b & -c & \cdots & -c \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -b & \cdots & a & -c & \cdots & -c \\ -c & \cdots & -c & a & \cdots & -c \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -c & \cdots & -c & -c & \cdots & a \end{pmatrix}, \tag{30}$$

whose inverse is given as a corollary below.

Corollary 4. *The inverse of the above A in (30) is given by*

$$A^{-1} = \begin{pmatrix} \frac{1}{a+b}I_{t\times t} & 0 \\ 0 & \frac{1}{a+c}I_{(n-t)\times(n-t)} \end{pmatrix} + \frac{1}{\omega_0}\begin{pmatrix} \frac{\omega_1}{a+b}E_{t\times t} & cE_{t\times(n-t)} \\ cE_{(n-t)\times t} & \frac{\omega_2}{a+c}E_{(n-t)\times(n-t)} \end{pmatrix}, \tag{31}$$

where $\omega_0 \neq 0$ is assumed, and ω_0, ω_1 and ω_2 are given by

$$\begin{aligned} \omega_0 &= [a - b(t-1)]\,[a - c(n-t-1)] - tc^2(n-t), \\ \omega_1 &= b\,[a - c(n-t-1)] + c^2(n-t), \\ \omega_2 &= c\,[a + b + t\,(c-b)]\,. \end{aligned} \tag{32}$$

Proof of Corollary 4. Given A in (30), (15) become:

$$\beta_1 = \frac{c}{a+b-t(b-c)}, \text{ and } \beta_i = \beta = \frac{c}{a+c} \text{ for } i = 2, \ldots, h = n-t+1;$$

$$\alpha = \frac{tc}{a+b-t(b-c)} + \frac{c(n-t)}{a+c},$$

$$\theta_1 = \frac{1}{1-\alpha}\left[\frac{tc}{a+b-t(b-c)} + \frac{(n-t)c^2}{b(a+c)}\right], \text{ and for } i = 2, \ldots, h,$$

$$\theta_i = \theta = \frac{1}{1-\alpha}\left\{\frac{c}{a+c} + \frac{c}{b}\left[\frac{tc}{a+b-t(b-c)} + \frac{c(n-t-1)}{a+c}\right]\right\}.$$

Substituting into (17), one gets: $A^{-1} =$

$$\frac{1}{a+b}I + \begin{pmatrix} \frac{b\beta_1(1+\theta_1)}{(a+b)c}E_{t\times t} & \frac{\beta_1}{(a+c)(1-\alpha)}E_{t\times 1} & \cdots & \frac{\beta_1}{(a+c)(1-\alpha)}E_{t\times 1} \\ \frac{\beta_1}{(a+c)(1-\alpha)}E_{1\times t} & \frac{b(1+\theta)}{(a+b)(a+c)} & \cdots & \frac{c}{(a+c)^2(1-\alpha)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_1}{(a+c)(1-\alpha)}E_{1\times t} & \frac{c}{(a+c)^2(1-\alpha)} & \cdots & \frac{b(1+\theta)}{(a+b)(a+c)} \end{pmatrix}.$$

Simplifying the above expressions with a, β_i and θ_i yields (31–32). □

Proof of Theorem 2. For $i = 1, \ldots, t, j = t + 1, \ldots, n$, the equations in (4) become: $\partial(\sum_{k\in S}\pi_k)/\partial p_i = 0$ and $\partial\pi_j/\partial p_j = 0$, which can be rearranged as

$$\begin{pmatrix} 2\delta & \cdots & -2\gamma & -\gamma & \cdots & -\gamma \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -2\gamma & \cdots & 2\delta & -\gamma & \cdots & -\gamma \\ -\gamma & \cdots & -\gamma & 2\delta & \cdots & -\gamma \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & \cdots & -\gamma & -\gamma & \cdots & 2\delta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} nV + \delta c_1 - \gamma\sum_{j=2}^{j=t} c_j \\ \vdots \\ nV + \delta c_t - \gamma\sum_{j=1}^{j=t-1} c_j \\ nV + \delta c_{t+1} \\ \vdots \\ nV + \delta c_n \end{pmatrix}, \tag{33}$$

where $\delta = n(1+\gamma) - \gamma$, $d_i = nV + \delta c_i - \gamma\Sigma_{j\in S\setminus i}c_j$, $i \in S$; $= nV + \delta c_i$, $i \notin S$. Note that (33) is actually (4) multiplied by n or $nAp = nd$. Solving for p by (31) with $a = 2\delta, b = 2\gamma, c = \gamma$, one gets $p^* = A^{-1}d =$

$$\begin{pmatrix} \frac{1}{2\delta+2\gamma}I_{t\times t} & 0 \\ 0 & \frac{1}{2\delta+\gamma}I_{(n-t)\times(n-t)} \end{pmatrix} d + \frac{1}{\omega_0}\begin{pmatrix} \frac{\omega_1}{2\delta+2\gamma}E_{t\times t} & \gamma E_{t\times(n-t)} \\ \gamma E_{(n-t)\times t} & \frac{\omega_2}{2\delta+\gamma}E_{(n-t)\times(n-t)} \end{pmatrix} d,$$

where $\omega_0 = \gamma^2\,(n-t)\,(t+2n-2) + 2n\gamma\,(3n-t-1) + 4n^2, \omega_1 = \gamma^2\,(t+3n-2) + 4n\gamma$ and $\omega_2 = 2n\gamma(1+\gamma) - t\gamma^2$. Substituting ω_1, ω_2 and $\delta = n(1+\gamma) - \gamma$ into above equations, one has: $p^* = A^{-1}d =$

$$\begin{pmatrix} \frac{1}{2n(\gamma+1)}I_t & 0 \\ 0 & \frac{1}{2n(1+\gamma)-\gamma}I_{n-t} \end{pmatrix} d + \begin{pmatrix} \frac{\gamma(n(4+3\gamma)+(t-2)\gamma)}{2n(\gamma+1)\omega_0}E_{t\times t} & \frac{\gamma}{\omega_0}E_{t\times(n-t)} \\ \frac{\gamma}{\omega_0}E_{(n-t)\times t} & \frac{\gamma(2n(1+\gamma)-t\gamma)}{(2n(1+\gamma)-\gamma)\omega_0}E_{(n-t)\times(n-t)} \end{pmatrix} d.$$

By collecting terms for V, c_i, $\sum_1^t c_k$ and $\sum_{t+1}^n c_k$, one gets (19). □

Proof of Theorem 3. Using the Dixit demand in (9), our first order conditions can be rearranged as

$$\begin{pmatrix} 2 & \cdots & -2\gamma & -\gamma & \cdots & -\gamma \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -2\gamma & \cdots & 2 & -\gamma & \cdots & -\gamma \\ -\gamma & \cdots & -\gamma & 2 & \cdots & -\gamma \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & \cdots & -\gamma & -\gamma & \cdots & 2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} V + (1+\gamma)c_1 - \gamma\sum_{j=1}^{j=t} c_j \\ \vdots \\ V + (1+\gamma)c_t - \gamma\sum_{j=1}^{j=t} c_j \\ V + c_{t+1} \\ \vdots \\ V + c_n \end{pmatrix}.$$

Solving for p by (31) with $a=2, b=2\gamma, c=\gamma; d_i = V+(1+\gamma)c_i - \gamma\Sigma_{j\in S}c_j$, $i\in S$, and $d_i = V+c_i$, $i\notin S$, one gets[9] $p=A^{-1}d=$

$$\begin{pmatrix} \frac{1}{2(1+\gamma)}I_{t\times t} & 0 \\ 0 & \frac{1}{2+\gamma}I_{(n-t)\times(n-t)} \end{pmatrix} d + \frac{1}{\omega_0^D}\begin{pmatrix} \frac{\omega_1^D}{2(1+\gamma)}E_{t\times t} & \gamma E_{t\times(n-t)} \\ \gamma E_{(n-t)\times t} & \frac{\omega_2^D}{2+\gamma}E_{(n-t)\times(n-t)} \end{pmatrix} d =$$

$$\begin{pmatrix} \frac{1}{2(1+\gamma)}I_{t\times t} & 0 \\ 0 & \frac{1}{2+\gamma}I_{(n-t)\times(n-t)} \end{pmatrix} d + \begin{pmatrix} \frac{-(n-2-t)\gamma^2+4\gamma}{2(1+\gamma)\omega_0^D}E_{t\times t} & \frac{\gamma}{\omega_0^D}E_{t\times(n-t)} \\ \frac{\gamma}{\omega_0^D}E_{(n-t)\times t} & \frac{(2-t)\gamma^2+2\gamma}{(2+\gamma)\omega_0^D}E_{(n-t)\times(n-t)} \end{pmatrix} d,$$

where $\omega_0^D = (t(n-t)-2(n-1))\,\gamma^2 - 2\,(n+t-3)\,\gamma + 4$, $\omega_1^D = -\,(n-2-t)\,\gamma^2 + 4\gamma$ and $\omega_2^D = (2-t)\,\gamma^2 + 2\gamma$. By collecting terms for V, c_i, $\sum_1^t c_k$ and $\sum_{t+1}^n c_k$, one gets (21). $\square$

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[9] By $q_k = V - p_k + \gamma\,\Sigma_{m\neq k}p_m = (1+\gamma-n\gamma)\left[\frac{V}{1+\gamma-n\gamma} - p_k - \frac{n\gamma(p_k-\bar{p})}{(1+\gamma-n\gamma)}\right]$, (21–22) can alternatively be obtained from (19–20) in two steps: replace V by $\widetilde{V}$ and γ by $\widetilde{\gamma}$ in (19–20) and then set $\widetilde{V} = V/\,(1+\gamma-n\gamma)$ and $\widetilde{\gamma} = n\gamma/(1+\gamma-n\gamma)$.